Chapter 2
Introduction to Binary Systems

In order to model stars, we must first have a knowledge of their physical properties. In this chapter, we describe how we know the stellar properties that stellar models are meant to replicate. Some of our data comes from observations of nearby single stars, but much of our information comes from binary stars. We will begin by describing the orbit of a binary and how these orbits are observed. We conclude this chapter with a discussion of how stellar masses are obtained from observations of the spectra of binary stars.

Binary systems are observed as:

1. *Visual* or *astrometric* binaries, if both or one of the stars can be observed to move in a periodic fashion
2. *Spectrum* or *spectroscopic* binaries if there are one or two clearly identified spectra indicating different Doppler shifts. Spectroscopic binaries have sufficiently short orbital periods so that a changing Doppler shift can be measured
3. *Eclipsing* binaries if the light from the system is observed to vary periodically as each star is eclipsed by its companion

Note that a given binary can be placed in more than one of these classifications.

In principal, the masses of the components of a binary can be inferred from a measurement of its orbital properties.

2.1 The Two-Body Problem

Given a central force, the motion of two bodies is found from the Lagrangian, which can be expressed as

$$\mathscr{L} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 + \frac{Gm_1 m_2}{|\mathbf{r}_2 - \mathbf{r}_1|}. \tag{2.1}$$

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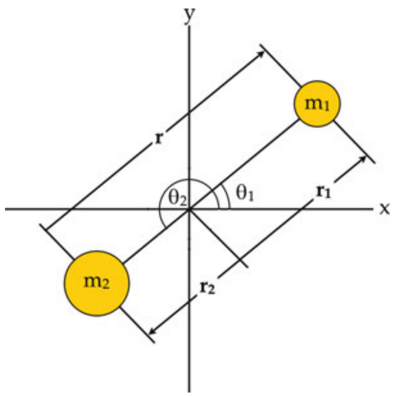


Fig. 2.1 Barycenter coordinate description of a binary system

We choose a barycentric coordinate system, so that

$$m_1\mathbf{r}_1 + m_2\mathbf{r}_2 = 0 \tag{2.2}$$

and therefore

$$m_1 r_1 = m_2 r_2. \tag{2.3}$$

We define the relative separation to be

$$r = r_1 + r_2. \tag{2.4}$$

We can use these two equations to solve for r_1 and r_2 in terms of r to get

$$r_1 = \frac{m_2}{M} r, \tag{2.5}$$

$$r_2 = \frac{m_1}{M} r, \tag{2.6}$$

where $M = m_1 + m_2$. Note that $\theta_1 = \theta_2 - \pi = \theta$ (Fig. 2.1).

Assuming that the orbits lie in a plane, we have

$$v_1^2 = \dot{r}_1^2 + r_1^2\dot{\theta}_1^2 = \left(\frac{m_2}{M}\right)^2 \left(\dot{r}^2 + r^2\dot{\theta}^2\right), \tag{2.7}$$

$$v_2^2 = \dot{r}_2^2 + r_2^2\dot{\theta}_2^2 = \left(\frac{m_1}{M}\right)^2 \left(\dot{r}^2 + r^2\dot{\theta}^2\right) \tag{2.8}$$

and so

$$\begin{aligned}
\mathscr{L} &= \frac{1}{2}\frac{m_1 m_2^2}{M^2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}\frac{m_2 m_1^2}{M^2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{G m_1 m_2}{r} \\
&= \frac{1}{2}\frac{m_1 m_2}{M}\dot{r}^2 + \frac{1}{2}\frac{m_1 m_2}{M} r^2\dot{\theta}^2 + \frac{G m_1 m_2 M}{M r} \\
&= \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\mu r^2\dot{\theta}^2 + \frac{G\mu M}{r}.
\end{aligned} \tag{2.9}$$

> **Problem 2.1:** Demonstrate that the orbit lies in a plane by obtaining the Lagrangian using arbitrarily oriented spherical polar coordinates (r, ϕ, θ). Calculate the Euler–Lagrange equations of motion and show that one can recover the planar equations of motion using the initial conditions: $\theta = \pi/2$ and $\dot{\theta} = 0$.

Since $\mathscr{L}$ is independent of θ, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\theta}} - \frac{\partial \mathscr{L}}{\partial \theta} \Rightarrow \frac{\partial \mathscr{L}}{\partial \dot{\theta}} = \text{constant} \tag{2.10}$$

so

$$\frac{\partial \mathscr{L}}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = J = \text{angular momentum.} \tag{2.11}$$

The total energy is also conserved, and it is given by

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{G m_1 m_2}{r} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{G\mu M}{r} = C. \tag{2.12}$$

(Note that here we use C for the total energy instead of E—this is because E is reserved for the *eccentric anomaly*, which is an important quantity for describing observations of orbits.) Using Eq. (2.11), we can express the total energy as an equation that is dependent upon r only.

$$\dot{\theta} = \frac{J}{\mu r^2} \Rightarrow \dot{\theta}^2 = \frac{J^2}{\mu^2 r^4}, \tag{2.13}$$

so

$$C = \frac{1}{2}\mu \dot{r}^2 + \frac{1}{2}\frac{J^2}{\mu r^2} - \frac{G\mu M}{r}. \tag{2.14}$$

2.2 The Orbital Shape

From Eq. (2.14), we can obtain the time dependence of the radius of the orbit, and then we can obtain the time dependence of the orbital angle using Eq. (2.11). However, these results are not particularly useful for determining the orbit directly from observations of binaries. Instead, we will first determine the shape of the orbit using Eq. (2.14) and some clever variable substitutions. Later we will determine the time dependence of the orbit in terms of observational quantities.

In order to determine the shape of the orbit, we first make the variable substitution $u = 1/r$, so that

$$\frac{\mathrm{d}u}{\mathrm{d}\theta} = u' = -\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta} \Rightarrow \frac{\mathrm{d}r}{\mathrm{d}\theta} = -r^2 u'. \tag{2.15}$$

Now,

$$\dot{r} = \frac{\mathrm{d}r}{\mathrm{d}\theta}\dot{\theta} = -r^2 u' \frac{J}{\mu r^2} = -\frac{J}{\mu}u', \tag{2.16}$$

where the $\dot{\theta}$ substitution comes from Eq. (2.13). Substitution of Eq. (2.16) into Eq. (2.14) gives

$$\frac{J^2}{2\mu}u'^{\,2} + \frac{J^2}{2\mu}u^2 - G\mu M u = C. \tag{2.17}$$

Now, we make another substitution and let $\ell = J^2/G\mu^2 M$ so that $J^2/\mu = GM\mu\ell$, and

$$\frac{1}{2}GM\mu\ell u'^{\,2} + \frac{1}{2}GM\mu\ell u^2 - GM\mu u = C. \tag{2.18}$$

Finally, we divide by $GM\mu/2\ell$ and add 1 to both sides to obtain

$$\ell^2 u'^{\,2} + \ell^2 u^2 - 2\ell u + 1 = \frac{2C\ell}{GM\mu} + 1. \tag{2.19}$$

Next, we define

$$e^2 = \frac{2C\ell}{GM\mu} + 1 \tag{2.20}$$

and make the final substitution of $x = \ell u - 1$, so we have

$$x'^{\,2} + x^2 = e^2, \tag{2.21}$$

or

$$x' = \sqrt{e^2 - x^2}. \tag{2.22}$$

This equation can be integrated as follows:

$$\begin{aligned} \int_{x_0}^{x} \frac{\mathrm{d}x}{\sqrt{e^2 - x^2}} &= \int_{\theta_0}^{\theta} \mathrm{d}\theta, \\ \arcsin\left(\frac{x}{e}\right) - \arcsin\left(\frac{x_0}{e}\right) &= \theta - \theta_0. \end{aligned} \tag{2.23}$$

Clearly, $|x| \leq |e|$ in order for the arcsin to make any sense. We define $\theta_0 = 0$ and require $x(0) = \mathrm{e}$ to obtain

$$\arcsin\left(\frac{x(0)}{e}\right) - \arcsin\left(\frac{x_0}{e}\right) = 0 \;\Rightarrow\; \arcsin\left(\frac{x_0}{e}\right) = \arcsin 1 = \frac{\pi}{2}. \tag{2.24}$$

Thus,

$$\frac{x}{e} = \sin(\theta + \pi/2) = \cos\theta \;\Rightarrow\; x = e\cos\theta. \tag{2.25}$$

Reversing all the substitutions, we finally obtain

$$r = \frac{\ell}{(1 + e\cos\theta)}, \tag{2.26}$$

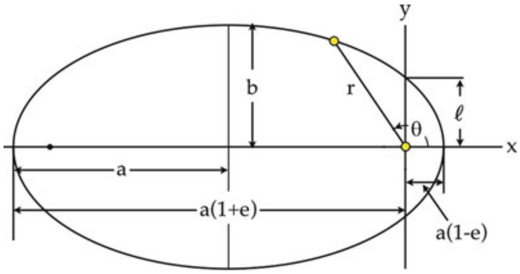


Fig. 2.2 Elliptical orbit with the origin centered on one star

which is the parametric equation for an ellipse. Thus, the shape of the relative orbit is an ellipse with the point of closest approach (or *periastron*) at $\theta = 0$ and one body at the focus. The semimajor axis (a) of an ellipse is half of the long axis, which is also the sum of the minimum distance and the maximum distance (the *apastron*). Thus,

$$r_{\min} = r(0) = \ell/(1+e), \tag{2.27}$$

$$r_{\max} = r(\pi) = \ell/(1-e) \tag{2.28}$$

and

$$a = \frac{1}{2}(r_{\min} + r_{\max}) = \ell/(1-e^2) \Rightarrow \ell = a(1-e^2). \tag{2.29}$$

The periastron and apastron can now be expressed in terms of the semimajor axis as

$$r_{\min} = a(1-e), \tag{2.30}$$

$$r_{\max} = a(1+e). \tag{2.31}$$

Although initially introduced to simplify the differential equation, the value of e is found to be the *eccentricity* of the elliptical orbit (Fig. 2.2).

Problem 2.2: Derive Kepler's third law ($GM = a^3\omega^2$) using $J = \mu r^2\dot{\theta}$ and $r = \ell/(1+e\cos\theta)$.

The actual motion of the components of the binary are about the center of mass (also known as the *barycenter*). We can show that this motion is also elliptical and obeys a version of Kepler's third law. Using barycentric coordinates, we have $m_1\mathbf{r}_1 = -m_2\mathbf{r}_2$ and $\mathbf{r}_1 - \mathbf{r}_2 = \mathbf{r}$. Therefore, from Newton's law, we have

$$\ddot{\mathbf{r}}_1 = -\frac{Gm_2}{r^3}\mathbf{r} = -\frac{Gm_2}{r_1^3}\left(\frac{m_2}{M}\right)^3\mathbf{r}_1\left(\frac{M}{m_2}\right) = -\frac{Gm_2^3}{M^2}\frac{\mathbf{r}_1}{r_1^3}. \tag{2.32}$$

We can obtain a similar equation for the motion of m_2 by simply interchanging 1 and 2. Note that these equations of motion are similar to the relative equation:

$$\ddot{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}, \tag{2.33}$$

$$\ddot{\mathbf{r}}_1 = -\frac{G\left(m_2^3/M^2\right)}{r_1^3}\mathbf{r}_1, \tag{2.34}$$

$$\ddot{\mathbf{r}}_2 = -\frac{G\left(m_1^3/M^2\right)}{r_2^3}\mathbf{r}_2, \tag{2.35}$$

and so they all obey a version of Kepler's third law with the following values for the mass:

Relative: M
Barycentric body 1: m_2^3/M
Barycentric body 2: m_1^3/M

Note also that there is a simple rescaling of the position vectors between the barycentric frame and the relative orbit frame:

$$\mathbf{r_1} = \frac{m_2}{M}\mathbf{r}. \tag{2.36}$$

$$\mathbf{r_2} = -\frac{m_1}{M}\mathbf{r}, \tag{2.37}$$

and so the barycentric orbits are simply rescaled versions of the relative orbit ellipse.

2.3 Time-Dependent Orbits

The orbital shape of the barycentric orbits is of value when we can only observe one star in the binary system. If we see both stars and can identify the motion of the barycenter, then we can identify the individual masses of the stars. Frequently, we only measure part of the orbit, and often we only measure the orbital speed. Thus, we need to know the position of the components as a function of time. This is found from what is known as *Kepler's equation*. To derive this we need to study the geometry of an ellipse.

Consider an ellipse with semimajor axis a that is circumscribed by a circle of radius a, as shown in Fig. 2.3.

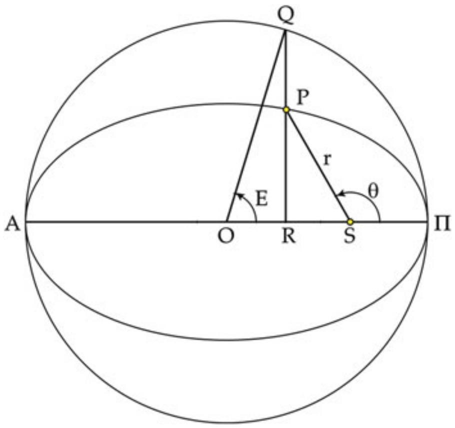
Referring to the figure, the following line segments and angles can be defined:

$$O\Pi = a = \text{ semimajor axis }, \tag{2.38}$$

$$S\Pi = a(1-e) = \text{periastron}, \tag{2.39}$$

$$OS = ae \tag{2.40}$$

Fig. 2.3 Properties of an ellipse



and

- The angle θ is called the *true anomaly*.
- The angle E is called the *eccentric anomaly*.

We want to find the time dependence of the eccentric anomaly, E.

The *auxiliary circle* has the property that $PR/QR = b/a = \sqrt{1-e^2}$, so

$$r\cos\theta \quad = -RS = OS - OR \quad = a\cos E - a\,e, \tag{2.41}$$

$$r\sin\theta = PR = \left(\sqrt{1-e^2}\right) QR = a\sin E\sqrt{1-e^2}, \tag{2.42}$$

and

$$\begin{aligned} r &= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta} \\ &= \sqrt{a^2e^2 - 2a^2e\cos E + a^2\cos^2 E + a^2\sin^2 E - a^2e^2\sin^2 E} \\ &= \sqrt{a^2e^2\left(1-\sin^2 E\right) - 2a^2e\cos E + a^2} \\ &= a\sqrt{e^2\cos^2 E - 2e\cos E + 1} \\ &= a\left(1 - e\cos E\right). \end{aligned} \tag{2.43}$$

We use the equation for the *specific angular momentum*, or angular momentum per mass:

$$r^2 \mathrm{d}\theta = L\mathrm{d}t \tag{2.44}$$

(n.b.: $L = J/\mu$), so we can substitute $r = a\left(1 - e\cos E\right)$, but we still need an equation for θ.

We obtain this equation by noting that

$$\frac{\mathrm{d}}{\mathrm{d}E}\sin\theta = \cos\theta\frac{\mathrm{d}\theta}{\mathrm{d}E}. \tag{2.45}$$

Using

$$\begin{aligned}\sin\theta &= \frac{a}{r}\sin E\sqrt{1-e^2}\\ &= \frac{a\sin E}{a\left(1-e\cos E\right)}\frac{b}{a}\\ &= \frac{b\sin E}{a\left(1-e\cos E\right)}\end{aligned} \tag{2.46}$$

and differentiating with respect to E gives

$$\frac{\mathrm{d}}{\mathrm{d}E}\left(\sin\theta\right) = \frac{b}{a}\frac{\cos E - e}{\left(1-e\cos E\right)^2} \tag{2.47}$$

so that

$$\cos\theta\mathrm{d}\theta = \frac{b\left(\cos E - e\right)\mathrm{d}E}{a\left(1-e\cos E\right)^2}. \tag{2.48}$$

Now, using

$$\cos\theta = \frac{-a\left(e-\cos E\right)}{r} = \frac{-\left(e-\cos E\right)}{\left(a-e\cos E\right)} \tag{2.49}$$

we find that

$$\mathrm{d}\theta = -\frac{\left(1-e\cos E\right)}{\left(e-\cos E\right)}\frac{b\left(\cos E - \mathrm{e}\right)}{a\left(1-e\cos E\right)^2}\mathrm{d}E = \frac{b\mathrm{d}E}{a\left(1-e\cos E\right)}. \tag{2.50}$$

Finally, we have

$$a^2\left(1-e\cos E\right)^2\frac{b\mathrm{d}E}{a\left(1-e\cos E\right)} = L\mathrm{d}t \tag{2.51}$$

or

$$\left(1-e\cos E\right)\mathrm{d}E = \frac{L}{ab}\mathrm{d}t. \tag{2.52}$$

Integrating this equation gives

$$\int\left(1-e\cos E\right)\mathrm{d}E = \frac{L}{ab}\int \mathrm{dt} \tag{2.53}$$

or

$$E - e\sin E = \frac{L}{ab}t + k. \tag{2.54}$$

Now we need to determine the integration constant. First, we define T to be the time at periastron passage and we note that $E = 0$ at periastron, so

$$k = -\frac{L}{ab}T \tag{2.55}$$

and

$$E - e\sin E = \frac{L}{ab}\,(t - T)\,. \tag{2.56}$$

From Kepler's second law, we have $\frac{1}{2}r^2\mathrm{d}\theta = \mathrm{d}A = \frac{1}{2}L\mathrm{d}t$, so

$$\int_0^{2\pi} \frac{1}{2}r^2\mathrm{d}\theta = \pi ab = \frac{1}{2}LP \;\Rightarrow\; \frac{L}{ab} = \frac{2\pi}{P} = \omega \tag{2.57}$$

and then

$$E - e\sin E = \frac{2\pi}{P}\,(t - T)\,. \tag{2.58}$$

This equation is generally solved using numerical techniques. The simplest approach is to use a Newton–Raphson iterative solution—given x_{n-1}, we find x_n by

$$x_n = x_{n-1} - f(x_{n-1})/f'(x_{n-1}). \tag{2.59}$$

Here, we let $f(E) = E - e\sin E - 2\pi(t-T)/P$ and note that $f'(E) = 1 - e\cos E$.

2.4 The Orbital Elements

Observed binaries do not lie in the plane of the sky, so we need to describe the orientation of the binary using the *orbital elements*. These are defined in terms of both the total angular momentum vector **J** and the total energy of the orbit.

The orientation of the binary can be described in terms of the direction of the total angular momentum vector and the direction of the periastron, which give the z- and x-axes in the orbital plane, respectively. These directions are measured relative to a coordinate system that is defined by the tangent plane to the celestial sphere at the location of the binary. A Cartesian coordinate system is defined in terms of the line of sight to the binary from the observer and the tangent to a great circle joining the binary to the north celestial pole. The angle of inclination is defined as the angle between the plane of the orbit and the tangent plane to the celestial sphere. The ascending node (N) is the line defined by the intersection of the plane of the orbit and the tangent plane and points in the direction where the binary passes from *inside* the celestial sphere to *outside* the celestial sphere. Figure 2.4 shows the orientation of the orbit relative to the tangent plane and the three angles that define this orientation. These three angles are

Angle of inclination i
Longitude of the ascending node Ω
Longitude of the periastron ω

The shape of the orbit is then given by three quantities:

Semimajor axis a
Eccentricity e
Time of periastron T

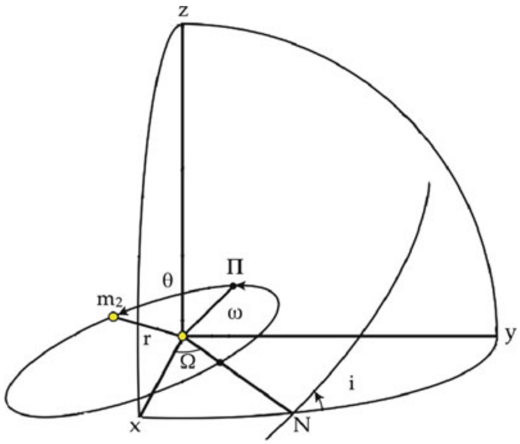


Fig. 2.4 Illustration of the different angles for the orbital elements of a binary. The line of sight from the observer to the binary is the along the z-axis, viewed from $z = -\infty$. The x-axis is chosen so that the positive x direction points toward the north celestial pole. The tangent plane of the sky is the xy-plane

These six quantities are called the orbital elements. If the orbital elements can be measured, then the masses of the binary can be determined. The orbit will always appear to be an ellipse when viewed on the sky, but unless $i = 0$, the center of mass of the system will not lie at the focus of this apparent ellipse (Fig. 2.5).

The angular momentum and total energy are also related to the orbital period and orbital shape. To obtain these relations we begin by noting that the kinetic energy is

$$K = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} \mu v^2, \tag{2.60}$$

where $v^2 = \dot{r}^2 + r^2\dot{\theta}^2$ and r and θ are relative separation variables. Now, using $r = \ell / (1 + e\cos\theta)$, we find that

$$\dot{r} = \dot{\theta}\frac{r^2}{\ell}\mathrm{e}\sin\theta = \frac{L}{\ell} e\sin\theta \tag{2.61}$$

and

$$r\dot{\theta} = \frac{r^2\dot{\theta}}{r} = \frac{L}{r} = \frac{L}{\ell}(1 + e\cos\theta)\,. \tag{2.62}$$

From here we get

$$\begin{aligned} v^2 &= \left(\frac{L}{\ell}\right)^2 \left[e^2\sin^2\theta + 1 + 2e\cos\theta + e^2\cos^2\theta\right] \\ &= \left(\frac{L}{\ell}\right)^2 \left[e^2 + 1 + 2e\cos\theta\right] = \left(\frac{L}{\ell}\right)^2 \left[2(1+\cos\theta) - 1 + e^2\right] \\ &= \left(\frac{L}{\ell}\right)^2 \left[\frac{2\ell}{r} - \left(1 - e^2\right)\right] = \frac{L^2}{\ell}\left[\frac{\ell}{r} - \frac{1-e^2}{\ell}\right] \\ &= \frac{L^2}{\ell}\left[\frac{2}{r} - \frac{1}{a}\right], \end{aligned} \tag{2.63}$$

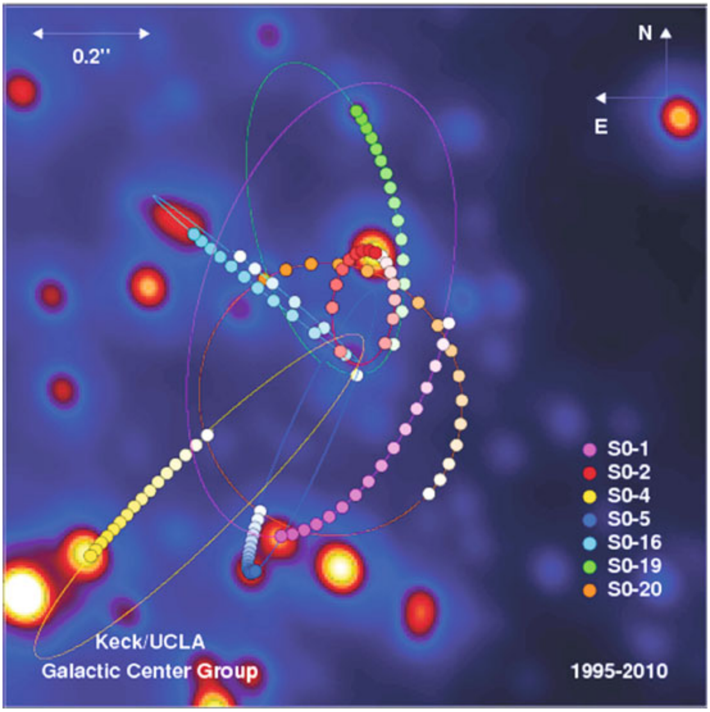


Fig. 2.5 Orbits of stars around Sgr A*. Note that every orbit is an ellipse, but that the foci do not all lie at a common point, even though all orbits are about the same object. This image was created by Prof. Andrea Ghez and her research team at UCLA and is from data sets obtained with the W. M. Keck Telescopes

where we have used $a = \ell\left(1-e^2\right)$ in the last step. Now from Kepler's second law, we have $L = 2\pi ab/P$, where P is the orbital period. Noting that $b^2 = a^2(1-e^2)$ we find

$$\begin{aligned} L &= \frac{4\pi^2 a^2 b^2}{P^2} = \frac{4\pi^2 a^3}{P} a\left(1-e^2\right) \\ &= GMa\left(1-e^2\right) \\ &= GM\ell, \end{aligned} \tag{2.64}$$

where we have used Kepler's third law. Finally, we have

$$v^2 = GM\left[\frac{2}{r} - \frac{1}{a}\right], \tag{2.65}$$

and so the kinetic energy is

$$K = \frac{1}{2}\mu v^2 = \frac{1}{2}\frac{m_1 m_2}{M} GM \left[\frac{2}{r} - \frac{1}{a}\right]$$
$$= \frac{Gm_1 m_2}{r} - \frac{Gm_1 m_2}{2a}. \tag{2.66}$$

Now, the potential energy is $\Omega = -Gm_1 m_2/r$, so the total energy is

$$C = K + \Omega = -\frac{Gm_1 m_2}{2a}. \tag{2.67}$$

The total angular momentum is $J = m_1 L_1 + m_2 L_2$, where

$$L_1 = \frac{m_2^2}{M^2} L, \tag{2.68}$$

$$L_2 = \frac{m_1^2}{M^2} L, \tag{2.69}$$

$$L^2 = GMa\left(1 - e^2\right). \tag{2.70}$$

This gives:

$$J = \frac{1}{M^2}\left(m_1 m_2^2 + m_2 m_1^2\right)\sqrt{GMa\left(1 - e^2\right)}$$
$$= m_1 m_2 \sqrt{\frac{Ga\left(1 - e^2\right)}{M}}$$
$$= \frac{2\pi}{P}\frac{m_1 m_2 a^2 \sqrt{1 - e^2}}{M}. \tag{2.71}$$

Thus, the total energy is fixed by the masses and the semimajor axis, while the total angular momentum also depends upon the period and the eccentricity.

2.5 Spectroscopic Binaries

We now look at determining the mass from spectroscopic binaries, where we can only measure the radial velocity of the component stars. The Doppler shift alters the frequency of spectral lines in stars by

$$f' = f\sqrt{\frac{c \pm v}{c \mp v}}, \tag{2.72}$$

where v is the radial velocity of the star and the sign choice depends on whether the star is moving toward us or away from us. From the frequency shifts, we can determine the total radial velocity which is a combination of the systemic motion of the binary and the velocity of the individual stars, so

$$v_{\text{rad}} = \dot{z} + \gamma. \tag{2.73}$$

From the orbital elements, we see that the z-component of the star in its orbit is given by

$$z = r\sin(\theta+\omega)\sin i, \tag{2.74}$$

and so the radial velocity is

$$\dot{z} = \sin i\left[\dot{r}\sin(\theta+\omega) + r\dot{\theta}\cos(\theta+\omega)\right]. \tag{2.75}$$

Since $r = a\left(1-e^2\right)/(1+e\cos\theta)$, we have

$$\dot{r} = er\dot{\theta}\sin\theta/(1+e\cos\theta). \tag{2.76}$$

Also, we have $r^2\dot{\theta} = 2\pi a^2\sqrt{1-e^2}/P$, and so

$$r\dot{\theta} = 2\pi a^2\sqrt{1-e^2}/rP = \frac{2\pi a(1+e\cos\theta)}{P\sqrt{1-e^2}}. \tag{2.77}$$

Substituting these two equations into Eq. (2.75), we find

$$\dot{z} = \frac{2\pi a\sin i}{P\sqrt{1-e^2}}\left[\cos(\theta+\omega) + e\cos\omega\right], \tag{2.78}$$

and so the total measured radial velocity is

$$v_{\text{rad}} = K\left[\cos(\theta+\omega) + e\cos\omega\right] + \gamma, \tag{2.79}$$

where $K = (2\pi a\sin i)/\left(P\sqrt{1-e^2}\right)$ is the *semi-amplitude of the velocity* and γ is the radial velocity of the center of mass. Note that K is not to be confused with the kinetic energy described in the previous section. A remarkable consequence of this result is that the extrema of v_{rad} are at the line of nodes. Several velocity curves for a variety of binary systems are shown in Fig. 2.6.

We can determine the value of K observationally by measuring the maximum and minimum velocities through the Doppler shift of spectral lines. Note that these values occur at $\theta+\omega = 0$ and π, respectively. Therefore,

$$v_{\max} = K\left[e\cos\omega + 1\right] + \gamma, \tag{2.80}$$

$$v_{\min} = K\left[e\cos\omega - 1\right] + \gamma, \tag{2.81}$$

and so

$$v_{\max} - v_{\min} = 2K \tag{2.82}$$

or

$$K = \frac{1}{2}\left(v_{\max} - v_{\min}\right). \tag{2.83}$$

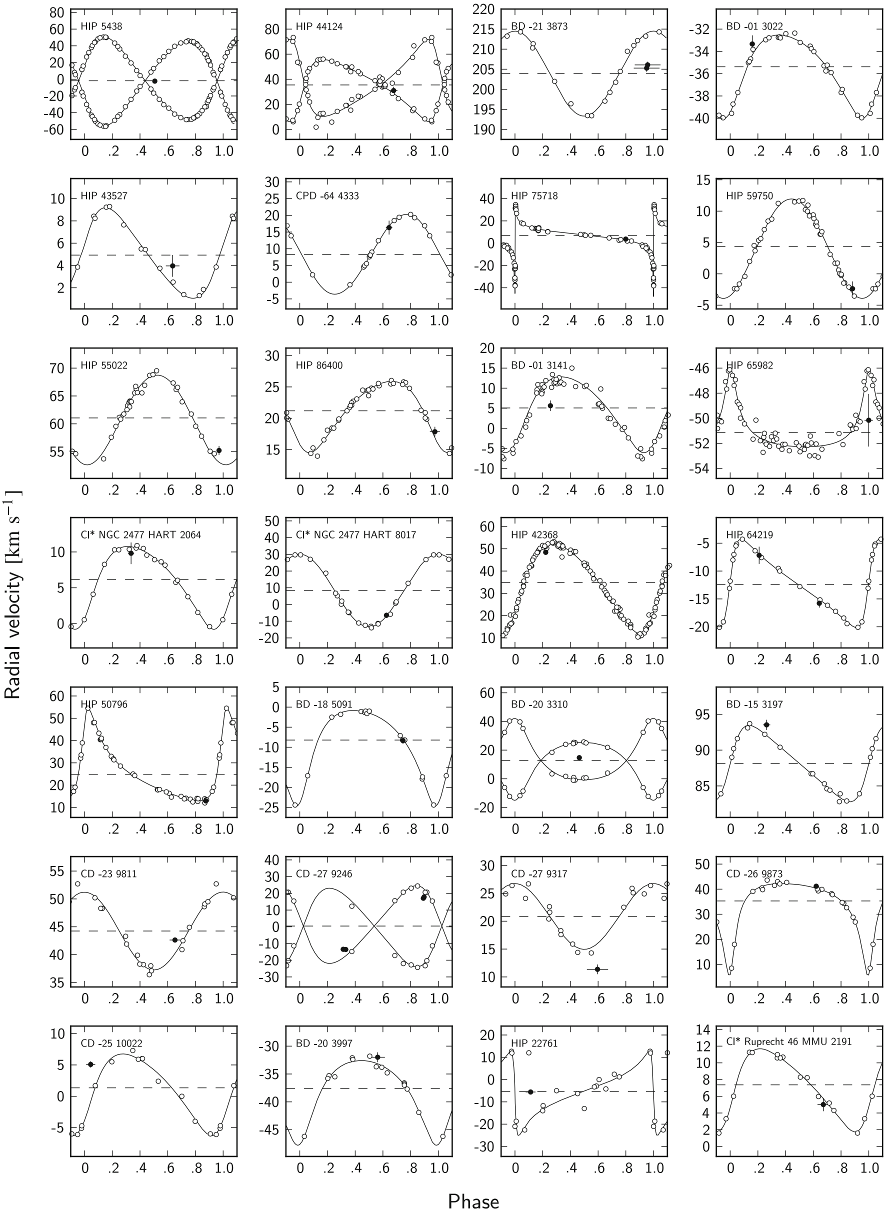


Fig. 2.6 Various velocity curves for several binary systems. Some are single-lined and some are double-lined. Figure taken from Matijevič, et al., *Astron. J.*, 141, 200 (2011). Reproduced by permission from the AAS

By fitting Eq. (2.75) to the shape of the velocity curve, one can obtain the values of e, ω, and γ. If a double-lined spectroscopic binary is observed, then we can determine

$$K_1 = \frac{2\pi a_1 \sin i}{P\sqrt{1-e^2}}, \tag{2.84}$$

$$K_2 = \frac{2\pi a_2 \sin i}{P\sqrt{1-e^2}} \tag{2.85}$$

along with e, ω, and γ. Therefore, we know

$$a_1 \sin i = \frac{\sqrt{1-e^2}}{2\pi} K_1 P, \tag{2.86}$$

$$a_2 \sin i = \frac{\sqrt{1-e^2}}{2\pi} K_2 P. \tag{2.87}$$

Since we know $m_1 a_1 = m_2 a_2$ and $GM = 4\pi^2 a^3/P^2$, we make the substitution:

$$m_2 = m_1 \left(a_1/a_2\right) = m_1 \left(\frac{a_1 \sin i}{a_2 \sin i}\right) = m_1 \left(K_1/K_2\right) \tag{2.88}$$

so

$$G m_1 \left(1 + \frac{K_1}{K_2}\right) = \frac{4\pi^2}{P^2} \left(a_1 \sin i + a_2 \sin i\right)^3 / \sin^3 i \tag{2.89}$$

and

$$\begin{aligned} m_1 \sin^3 i &= \frac{4\pi^2}{P^2} \frac{K_2}{G\left(K_1 + K_2\right)} \left(\frac{\sqrt{1-e^2}}{2\pi} P\right)^3 \left(K_1 + K_2\right)^3 \\ &= \frac{P}{2\pi G} \left(1 - e^2\right)^{3/2} \left(K_1 + K_2\right)^2 K_2. \end{aligned} \tag{2.90}$$

This provides an upper limit for m_1 unless i is known. We can find an upper limit for m_2 by simply interchanging 1 and 2 in Eq. (2.90). If we can only measure the radial velocity of one component of the binary (say K_1), then we can determine the *mass function* by using Eq. (2.88) to determine K_2 in terms of m_1, m_2, and K_1. We substitute this expression for K_2 into Eq. (2.90) to obtain

$$m_2 \sin^3 i = \frac{P K_1^3}{2\pi G} \left(1 - e^2\right)^{3/2} \left(\frac{m_1 + m_2}{m_2}\right)^2, \tag{2.91}$$

and so

$$f(m) = \frac{m_2^3 \sin^3 i}{\left(m_1 + m_2\right)^2} = \frac{P K_1^3}{2\pi G} \left(1 - e^2\right)^{3/2}, \tag{2.92}$$

where $f(m)$ is known as the mass function.

If the orbit is also a visual binary, it is possible to obtain the angle of inclination and consequently to obtain exact values for m_1 and m_2. The direct measurement of the masses of all stars except the sun is determined in this way.

Problem 2.3: MT720 is a spectroscopic binary in the Cygnus OB2 Association. It is found to have a period of $P = 4.36$ d and an eccentricity of e $= 0.35$. The semi-amplitude of the radial velocities are $K_1 = 173\,\mathrm{km/s}$ and $K_2 = 242\,\mathrm{km/s}$.

(a) Find $m \sin^3 i$ and $a \sin i$ for each star.
(b) What is the mass ratio: $q = m_2/m_1$?
(c) If $i = 70°$, what are the masses of each star?

Problems

2.1. Demonstrate that the orbit lies in a plane by obtaining the Lagrangian using arbitrarily oriented spherical polar coordinates (r, ϕ, θ). Calculate the Euler–Lagrange equations of motion and show that one can recover the planar equations of motion using the initial conditions: $\theta = \pi/2$ and $\dot{\theta} = 0$.

2.2. Derive Kepler's third law ($GM = a^3\omega^2$) using $J = \mu r^2 \dot{\theta}$ and $r = \ell/(1 + e\cos\theta)$.

2.3. MT720 is a spectroscopic binary in the Cygnus OB2 Association. It is found to have a period of $P = 4.36$ d and an eccentricity of e $= 0.35$. The semi-amplitude of the radial velocities are $K_1 = 173\,\mathrm{km/s}$ and $K_2 = 242\,\mathrm{km/s}$.

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